(Riemann) Integration Sucks!!!

Peyam Ryan Tabrizian

Friday, November 18th, 2010

1 Are all functions integrable?

Unfortunately not! Look at the handout "Solutions to 5.2.67, 5.2.68", we get two examples of functions f which are not Riemann integrable!

Let's recap what those two examples were!
The first one was $f(x) = \frac{1}{x}$ on $[0,1]$. The reason that this function fails to be integrable is that it goes to ∞ is a very fast way when x goes to 0, so the area under the graph of this function is infinite.

Remember that it is not enough to say that this function has a vertical asymptote at $x = 0$. For example, the function $g(x) = \frac{1}{\sqrt{x}}$ has an asymptote at $x = 0$, but it is still integrable, because by the FTC:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_0^1 = 2$$

This is not *too* bad! Basically, we have that $f(x) = \frac{1}{x}$ is not integrable because its integral is 'infinite', whatever this might mean! The point is: if a function is integrable, then its integral is **finite**.

There's a way to get around this! In Math 1B, you'll learn that even though this function is not integrable, we can **define** its integral to be ∞, so you'd write:

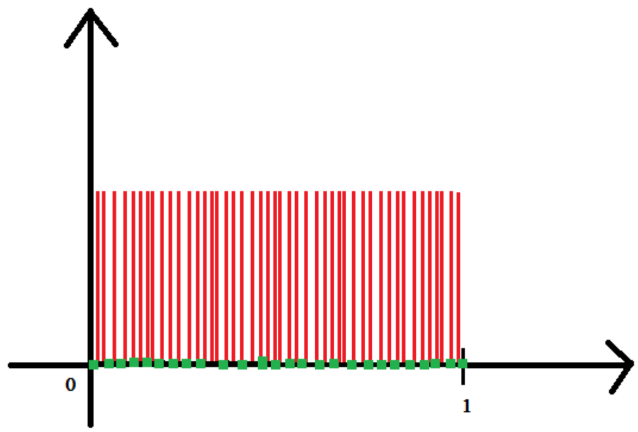
$$\int_0^1 \frac{1}{x} dx = \infty$$

The second example was way worse! The function in question was:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

And its graph looks like this:

1A/Handouts/Nonintegrable.png



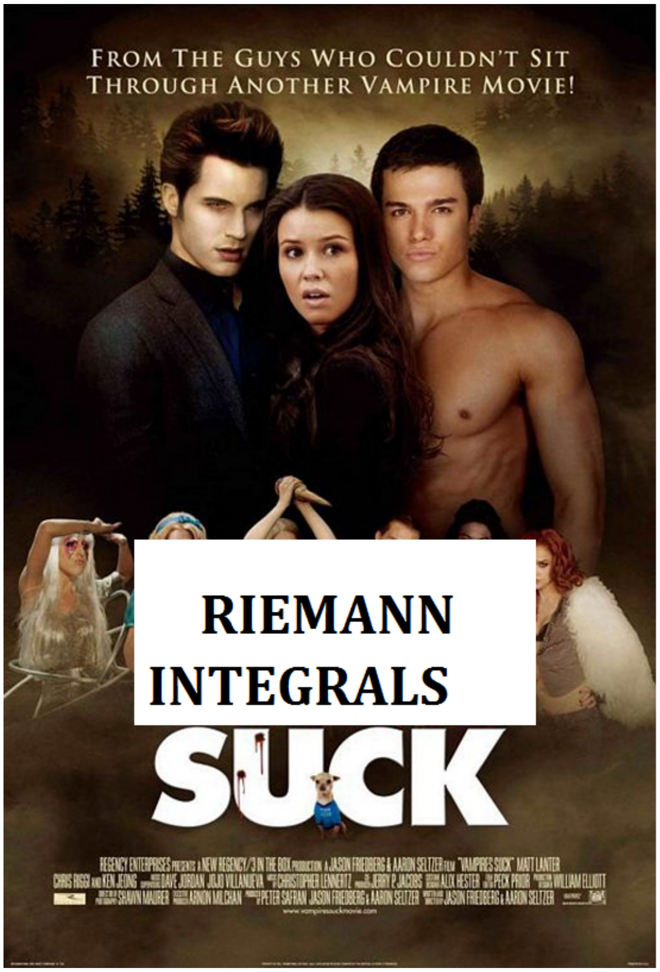
This function is not integrable because it basically can't make up its mind! It sometimes is 0 and sometimes is 1.

2 Oh no, what do we do?

What do we do with this second example? You might just sweep it under the rug and pretend it never existed. But believe it or not, you *do* use this function **a lot** in applications! One immediate example I can think of is: imagine you're in a room with a light switch, and think of f being equal to 1 when the light is on, and equal to 0 when the light is off. If you play with the switch like crazy, you get the graph above!

It's not really way the function behaves that is a problem, but rather the way we defined (Riemann) integration! Riemann integration is pretty good for functions we're dealing with in this class, but in real life...

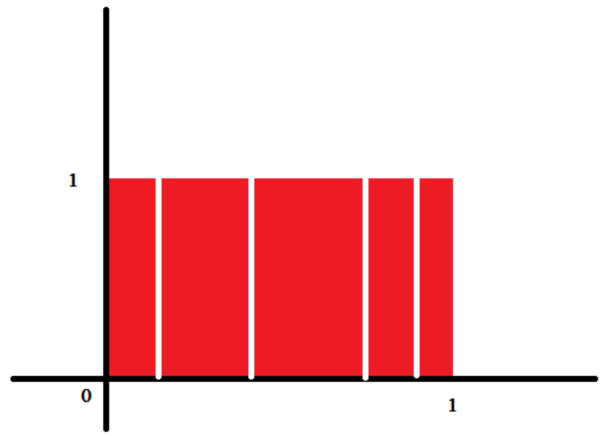
1A/Handouts/Integrals Suck.png



Even though Riemann integration fails, let's intuitively see what the area under the graph of this function should be. Notice the following: There are **MANY, MANY, MANY** more irrational numbers than rational numbers! In fact, if you pick a random number between 0 and 1, the probability that it's irrational is 1 !

So the graph of the above function should actually look more like this:

1A/Handouts/Nonintegrable2.png



(This is also a little bit misleading, since there are also a lot rational numbers, but it illustrates the point I'm going to make).

So looking at this second picture, even though f is not Riemann integrable, we would still like to say "The area under its graph is 1". This is because, on $[0, 1]$, $f(x) = 1$ *most* of the time!

And this is why we need to redefine our notion of integral! One which takes into account functions like the above!

3 Lebesgue Integration

So how do we go about defining a more powerful kind of integration? Not too long ago (the 1900's), a French mathematician called Henri Lebesgue came up with a brilliant idea! How about, *instead* of splitting up the interval $[0, 1]$ as we usually do with Riemann integrals, we split up the **range** of f? Let's see what happens if we apply this with f above! In this case, the range of f is very easy, it only consists of two points, 0 and 1! So we just need to look at two sets: A_0, which is the set of points where $f(x) = 0$, and A_1, which is the set of points where $f(x) = 1$. By definition of f, we get that A_0 is the set of rational numbers between 0 and 1, and A_1 is the set of irrational numbers between 0 and 1.

So you would simply define (don't worry about the 'dm'-part, it just means 'Lebesgue integral'):

$$\int_0^1 f(x)dm = 0 \cdot m(A_0) + 1 \cdot m(A_1)$$

Where $m(A_0)$ is the 'size' of A_0 and $m(A_1)$ is the 'size' of A_1. Of course we would have to be more precise with the notion of 'size', but I will spare you the details because it involves a **LOT** of $\epsilon - \delta$. However, intuitively, $m(A_0)$, which is the size of the rational numbers in $[0,1]$, should be 0, and $m(A_1)$, which is the size of irrational numbers in $[0,1]$ should be 1, just because, again, there are many more irrational numbers than rational numbers. And this actually agrees with the precise definition!

And hence, we get:

$$\int_0^1 f(x)dm = 0 \cdot 0 + 1 \cdot 1 = 1$$

Which is just what we wanted!!! So Lebesgue integration solved a problem that Riemann integration failed to solve!!!

In general, the definition of the Lebesgue integral is a bit more complicated, but looks a lot like the Riemann integral. A function f like the one above is called a **simple function** because its range only has a couple of points (here 0 and 1). For a simple function with range $\{y_1, y_2, \cdots, y_n\}$, we define its integral to be:

$$\int_a^b f(x)dm = \sum_{i=1}^{n} y_i \cdot m\left(A_{y_i}\right)$$

Where A_{y_i} is the set of points x in $[a,b]$ such that $f(x) = y_i$.

Now, for general functions f, we define the Lebesgue integral in the following way:

$$\int_a^b f(x)dm = \lim_{n\to\infty} \int_a^b f_n(x)dm$$

Where $\{f_n\}$ is *any* sequence of **simple** functions that approaches f as $n \to \infty$ (Think of the f_n sort of like the x_i^*, here again, notice that it should be true for **any** choice of f_n that satisfies those conditions!).

A natural question to ask is: When does this make sense? This leads to the notion of Lebesgue integrability:

Definition. *A function f is **Lebesgue integrable** if such a choice of $\{f_n\}$ exists, and if we get the same answer no matter how we choose our f_n!*

Notice how this definition of integrability is similar to our previous notion of Riemann integrability! And in fact, one can show that Riemann integrable functions are still Lebesgue integrable!

Finally, the question is: are all functions Lebesgue integrable? The sad answer is: **NO**. For example, $\frac{1}{x}$ is still not Lebesgue integrable, because its integral is still infinity. But the good news is that **A LOT** of functions that are not Riemann integrable, especially those which arise in applications, are Lebesgue integrable! If you want to cook up an example of a function (not like $\frac{1}{x}$) that is not Lebesgue integrable, you'd have to work very very very hard!

So again, to summarize, Lebesgue integration is a truly remarkable device, and should be considered the adult version of our current notion of integration, which, although pretty powerful, is too weak for advanced applications!

If you're interested in this, I urge you to take Math 202A, which is all about this stuff! You actually only need to take Math 104 for this, which technically only requires Math $1A$!!!

4 Application to probability

One thing that makes Lebesgue integration so awesome is that in probability theory, it unifies discrete and continuous random variables in a single integral.

If X has a probability density function f (this is called 'X is continuous'), then the expectation of X is just:

$$E(X) = \int_{-\infty}^{\infty} f(x)dx$$

But what if X is discrete and does not have a probability density function? Then, the definition of expectation is:

$$E(X) = \sum_{n=-\infty}^{\infty} nP(X = n)$$

But notice that $\{X = n\}$ is precisely the set where X takes value n, and n is that value! In other words, using the notation of the previous section, we can write $A_n = \{X = n\}$, and we get:

$$E(X) = \sum_{n=-\infty}^{\infty} nP(A_n)$$

But this is precisely the Lebesgue integral of X with respect to the measure $m = P$[1]

Hence we can write:

[1] At least if X has finite values, but one can show the same holds if X has infinite values

$$E(X) = \int_{-\infty}^{\infty} X dP$$

Notice the similarity with the continuous case! Hence, in general, probabilists just use the notation $\int X dP$ to denote $E(X)$, irregardless of whether X is continuous or discrete.

5 The Stieltjes integral

This has nothing to do with Lebesgue integration, but it's a nice generalization of Riemann integration. Recall the definition of the Riemann integral:

$$\int_a^b f(x)dx = n \overset{\lim}{\to} \infty \sum_{i=1}^{n} f\left(x_i^*\right)\left(x_i - x_{i-1}\right)$$

Where $\Delta x = \frac{b-a}{n}$, and $x_i = a + (\Delta x)\, i$, and $x_i^* \in [x_{i-1}, x_i]$

Then the Stieltjes integral (which depends on a fixed function α) is just:

$$\int_a^b f(x)d\alpha(x) = n \overset{\lim}{\to} \infty \sum_{i=1}^{n} f\left(x_i^*\right)\left(\alpha\left(x_i\right) - \alpha\left(x_{i-1}\right)\right)$$

This is useful when you want to put emphasis on a certain point, which you might find more statistically significant. For example, you could define $\alpha(x) = x$ if $x \neq 0$ and $\alpha(0) = 1000$ to emphasize that 0 is very important. Or you could define $\alpha(x) = e^x$ to say that later values of x are much more important.

6 Ito's integral

This integral is very useful in stochastic processes and nicely combines the ideas of the Lebesgue integral and the Stieltjes integral.

Definition. *$X = X(t)$ is a* ***stochastic process*** *if for every t, $X(t)$ is a random variable*

Just as in the Lebesgue integral, we need to define **simple** stochastic processes (which is the analog of simple functions):

Definition. *$X(t)$ is called a simple stochastic process if there exists a partition $\{t_0 = a, t_1, \cdots, t_n = b\}$ of $[a, b]$ and random 'values' $X_1, \cdots, X_n$ which* ***do not depend on*** *t such that $X(t) = X_i$ on each subinterval $[t_{i-1}, t_i]$.*

Then, if X is simple stochastic process with values X_i on $[t_{i-1}, t_i]$ and W is the standard Brownian motion (think of a particle that moves really randomly on the line), we can define the **Ito integral** of X as[2]:

[2]Notice how similar this is to the Stieltjes integral; in other words, $\alpha = W$ here!

$$\int_a^b X dW = \sum_{i=1}^{n} X_i \left(W\left(t_i\right) - W\left(t_{i-1}\right)\right)$$

Notice that this is a random variable, not a number!

And then, just as for the Lebesgue integral, if X is a *general* stochastic process, one can show that (if X is nice enough), there exists a sequence X^n of *simple* stochastic processes such that $\lim_{k\to\infty} X_k = X$, and then we can define the Ito integral as.

$$\int_a^b X dW = \lim_{k\to\infty} \int_a^b X_k dW$$

7 Stochastic Differential Equations

Using the Ito integral, we can now define Stochastic Differential Equations, which is a differential equation involving random variables (a 'random' differential equation). They are very useful in probability and in finance.

Namely, if F and G are functions on $\mathbb{R} \times [0, T]$ (**NOT** random variables), then we say that X (a real-valued random variable) satisfies the **stochastic differential equation**

$$dX = F(X, t)dt + G(X, t)dW$$

if, for all $0 \leq t \leq T$, we have:

$$X(t) = X(0) + \int_0^t F(X(t), t)dt + \int_0^t G(X(t), t)dW$$

In other words, just integrate the formula $dX = F(X, t)dt + G(X, t)dW$ with respect to t !

Notice that the second integral $\int_0^t F(X(t), t)dt$ is just a Lebesgue integral (which spits out a random variable), but $\int_0^t G(X(t), t)dW$ is an Ito integral.

8 The Peyam Integral

Does not exist yet, but stick around, it's gonna be awesome and revolutionize the way people think about integration :)